Indian Statistical Institute

Final Examination. 2024-2025

Analysis II, B.Math First Year

Time : 3 Hours   Date : 30.04.2025   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You may freely apply any of the theorems discussed in class.

Q1. (15 marks) Let r and s be the radii of convergence of the power series $\sum_{n=0}^{\infty} r_n x^n$ and $\sum_{n=0}^{\infty} s_n x^n$, respectively. If t is the radius of convergence of the power series $\sum_{n=0}^{\infty} r_n s_n x^n$, then prove that $t \geq rs$.

Q2. (15 marks) For each $n = 1, 2, \ldots$, define

$$f_n(x) = (-1)^n \frac{x^2 + n}{n^2} \qquad (x \in \mathbb{R}).$$

Prove that

$$\sum_{n=1}^{\infty} f_n(x),$$

converges uniformly on any bounded subset of $\mathbb{R}$ but does not converge absolutely for any $x \in \mathbb{R}$.

Q3. (15 marks) Does the improper integral

$$\int_1^{\infty} \frac{\cos \frac{1}{x}}{\sqrt{x}}\, dx$$

converge or diverge? Justify your answer.

Q4. (15 marks) Suppose the series of real numbers $\sum_{n=0}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-1, 1]$ to a continuous function.

Q5. (15 marks) (15 marks) Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Prove that

$$\lim_{x \to 1^-} \int_0^x f(t) dt = \int_0^1 f(t) dt.$$

Q6. (15 marks) Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ converging pointwise to a Riemann integrable function f on $[a, b]$. Prove that

$$\int_a^b f \leq \liminf_n \int_a^b f_n.$$

Q7. (20 marks) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that there exists a sequence of polynomials $\{p_n\}$ such that

$$f = \sum p_n,$$

converges uniformly on $[0, 1]$.